Discrete logarithms: Recent progress (and open problems)

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Discrete logarithms

- Given a multiplicative group G with generator g
- Computing discrete logarithms is inversing $n \longrightarrow g^n$
- Hard in general and used as a hard problem in cryptography
- Algorithmic viewpoint
  - Generic algorithms (for any G)
    - Pohlig-Hellman
    - Baby step, Giant step and Pollard's Rho
  - Specific algorithms (make use of group representation)

Classical groups for Dlog in Cryptography

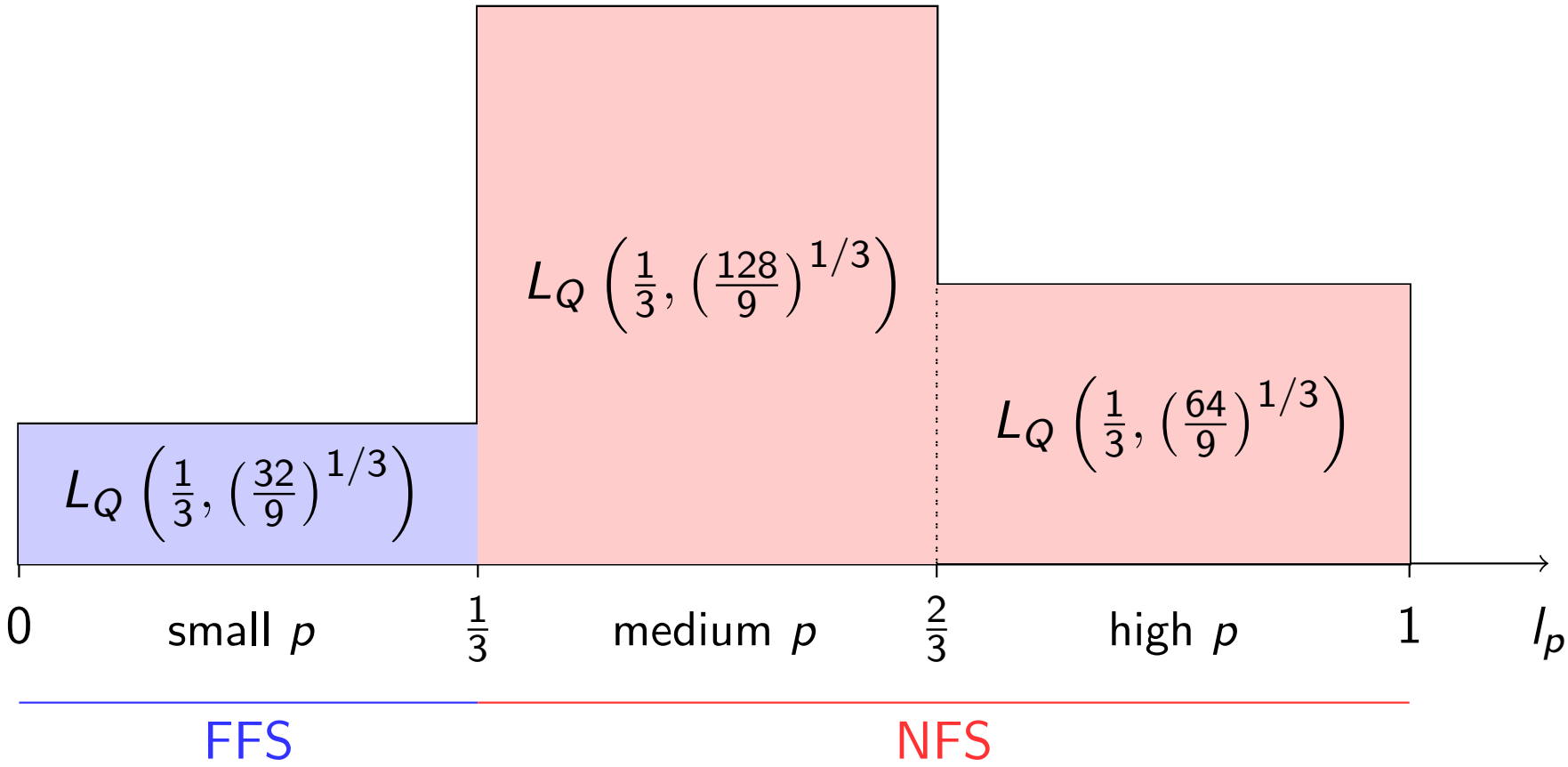
- Integers modulo p
- More general finite fields $\mathbb{F}_{p^k}$
- Elliptic curves over finite fields

Index calculus algorithms

- Relation generation phase
  - Generates many sparse equations
  - Modulo group order for discrete log
    (Modulo 2 for factoring)
- Linear algebra phase
  - Large sparse system
  - Numbers of unknowns in range up to dozens of millions
  - Number of equations potentially very large
  - Need to use large computers to solve such systems
- Individual logarithm phase

Complexity of Index calculus algorithms

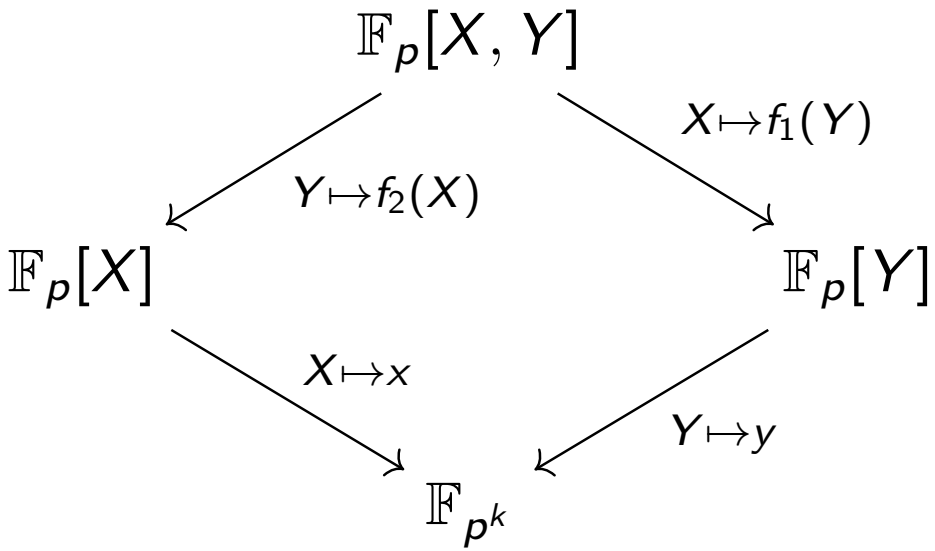
$$L_Q(\beta, c) = \exp((c + o(1))(\log Q)^{\beta}(\log\log Q)^{1-\beta}).$$



Discrete Logarithms in the Medium prime case [JL06]

- Finite field of the form $\mathbb{F}_{p^k}$
- Choose two univariate polynomials f_1 and f_2
  - with degrees d_1 and d_2 and $d_1 d_2 \geq k$.
  - Such that $x - f_1(f_2(x))$ has:
    - an irreducible factor of degree k (modulo p).
- This defines the finite field by the relations:
  - $x = f_1(y)$ and $y = f_2(x)$

Commutative diagram



Discrete Logarithms in the Medium prime case [JL06]

- Optimal for $p = L_{p^k}(1/3)$

- Choose smoothness basis $x - \alpha$ and $y - \alpha$
- Consider elements:

$$\begin{aligned} xy + ay + bx + c &= x\, f_2(x) + a f_2(x) + bx + c \\ &= y\, f_1(y) + ay + b f_1(y) + c \end{aligned}$$

- When both sides split $\Rightarrow$ Relation
- Heuristic cost of finding relation (sieving):

$$(d_1 + 1)!\,(d_2 + 1)!$$

- Individual log. descent negligible compared to initial phase

Nice special case – Kummer extensions

- Assume $k|p-1$, then $\mathbb{F}_{p^k}$ can be defined by $x^k - t$
- If $k = d_1 d_2 - 1$, let $y = x^{d_1}$ and $tx = y^{d_2}$ (*i.e.* $tx = x^{d_1 d_2}$)
- Reduces size of smoothness basis by k
  - Indeed:

$$\begin{aligned}(X+\alpha)^p &= X^p + \alpha = t^{(p-1)/k} X + \alpha = \mu(X + \alpha/\mu),\\ (Y+\alpha)^p &= \mu^{d_1}(Y + \alpha/\mu^{d_1}).\end{aligned}$$

    where μ is a k-th root of unity in $\mathbb{F}_p$.

- Can be generalized to $k = d_1 d_2 + 1$ using $y = x^{d_1}$ and $x = t/y^{d_2}$

Linear change of variables [J13]

- Further restrict to $y = x^{d_1}$
- Then:

$$xy + ay + bx + c \quad = \quad x^{d_1+1} + ax^{d_1} + bx + c$$

- Perform change of variable: $x = aX$, we get:

$$a^{d_1+1}(X^{d_1+1} + X^{d_1} + b \cdot a^{-d_1}(X + c/(ab)).$$

- Change of variable does not affect splitting property
- One good left-hand side $\Rightarrow$ p good left-hand sides
- Amortized cost of relation reduced to

$$\left(\frac{(d_1+1)!}{p-1} + 1\right) \cdot (d_2+1)!$$

Case of Kummer extensions

- Assume $k|p-1$, i.e. $\mathbb{F}_{p^k}$ can be defined by $x^k - t$
- If $k = d_1 d_2 - 1$, let $y = x^{d_1}$ and $tx = y^{d_2}$
  - $x^{d_1+1} + ax^{d_1} + bx + c \Rightarrow a^{d_1+1}(X^{d_1+1} + X^{d_1} + b \cdot a^{-d_1}(X + c/(ab))$.
  - $(y^{d_2+1} + by^{d_2})/t + ay + c \Rightarrow$
    $b^{d_2+1}\left((Y^{d_2+1} + Y^{d_2})/t + a \cdot b^{-d_2}(Y + c/(ab))\right)$.
- In both cases $\lambda = c/(ab)$ is shared by the two sides

Kummer extensions – Reassembling two sides

- Assume that:
  - $X^{d_1+1} + X^{d_1} + \theta_X(X + \lambda)$ splits and
  - $(Y^{d_2+1} + Y^{d_2})/t + \theta_Y(Y + \lambda)$ splits.
- Find a and b such that $\theta_X = b \cdot a^{-d_1}$ and $\theta_Y = a \cdot b^{-d_2}$?
- This implies $\theta_X^{d_2}\theta_Y = a^{-d_1 d_2+1} = a^{-k}$.
  - Possible iff $\theta_X^{d_2}\theta_Y$ is a k-th power
  - Gives k (conjugate) solutions !
  - From a recover b and c
  - Roots obtained by change of variable

Impact in the medium prime case

- In theory, reduces constant in $L(1/3)$ complexity of function field sieve.

- In practice, Kummer extensions esp. good for records:
  - First 1175-bit field $\mathbb{F}_{p^{47}}$ with p close to 2^{25}
  - Then 1425-bit field $\mathbb{F}_{p^{57}}$ with p close to 2^{25}
  - Previous finite field record was 923 bits
  - Timings: about 32000 CPU-hours compared to 895000 CPU-hours

- $47 = 6 \cdot 8 - 1$
- $57 = 7 \cdot 8 + 1$

Small characteristic – Setting [J13b]

- Define finite field by a relation:

$$x^{p^\ell} = \frac{h_0(x)}{h_1(x)},$$

  gives degree $k = \deg(I(x))$ extension, where $I(x)$ is a divisor of $h_1(x)x^{p^\ell} - h_0(x)$.
- We have a systematic relation:

$$x^{p^\ell} - x = \prod_{\alpha \in \mathbb{F}_{p^\ell}} (x - \alpha).$$

Small characteristic – Basic idea [J13b]

- Use more general change of variable: $x = \frac{aX+b}{cX+d}$, we get:

$$(cX+d)\cdot(aX+b)^{p^\ell} - (aX+b)\cdot(cX+d)^{p^\ell} =$$
$$(cX+d)\cdot\prod_{\alpha\in\mathbb{F}_{p^\ell}}((a-\alpha\, c)X+(b-\alpha\, d))$$

- Moreover, after expanding the left-hand size, we find:

$$(ca^q-ac^q)X^{q+1}+(da^q-bc^q)X^q+(cb^q-ad^q)X+(db^q-bd^q),$$

  where $q = p^\ell$.
  It becomes a low degree polynomial after multiplying by h_1 and replacing $h_1(X)\,X^q$.
- As a consequence, multiplicative relations are very easy to find

Small characteristic – Choice of a, b, c and d

- If a, b, c and d are in $\mathbb{F}_q$ left-hand side is:

$$(ad - bc)(X^q - X) \Rightarrow \text{Trivial relation}$$

- Take a, b, c and d in small extension field such as $\mathbb{F}_{q^2}$
- Some choices of (a, b, c, d) are equivalent. Good parametrization is:

$$PGL_2(\mathbb{F}_{q^2})/PGL_2(\mathbb{F}_q)$$

Small characteristic – Resulting Complexity [J13b]

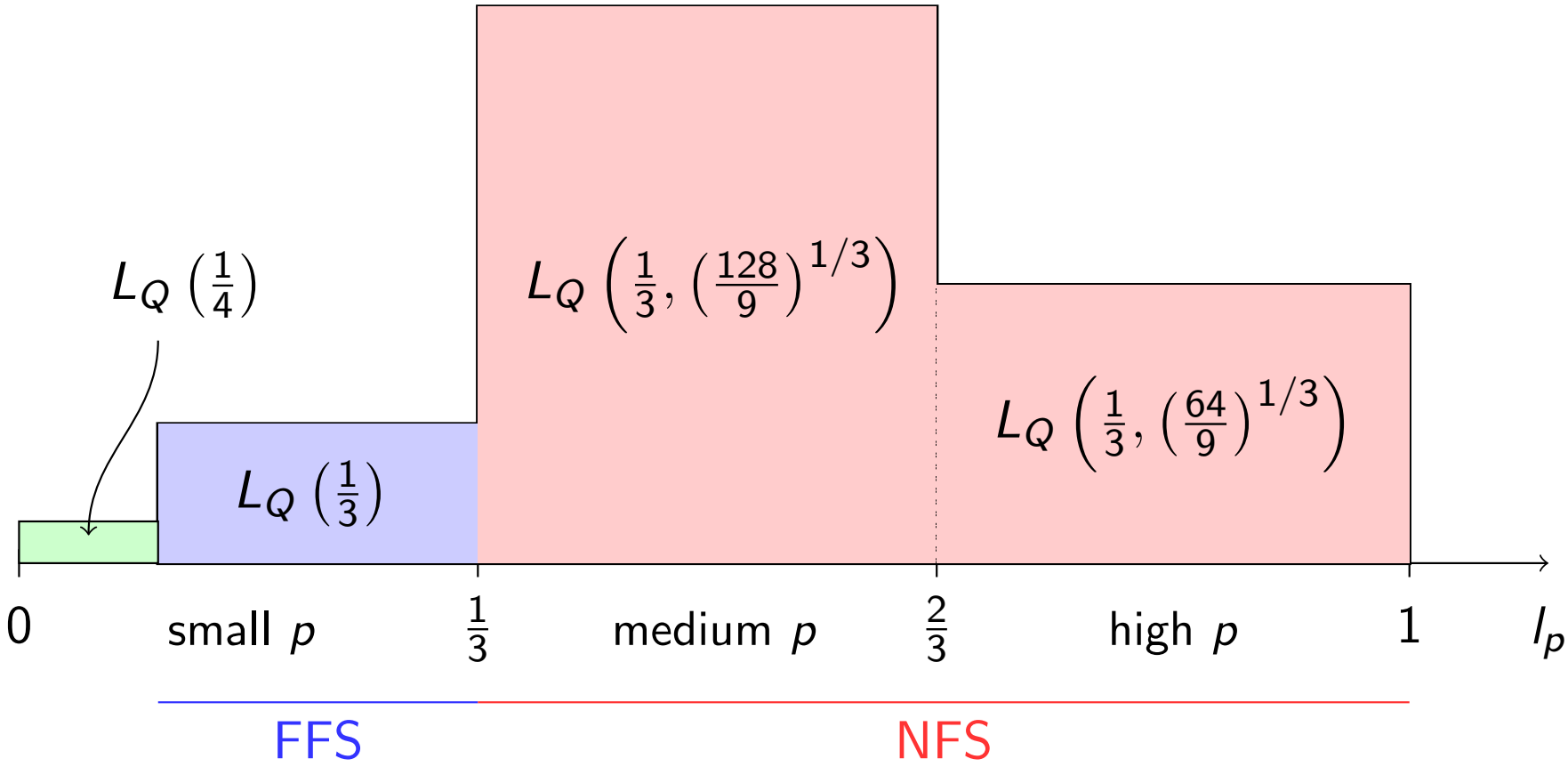
- Logarithms of smoothness basis in polynomials time
  - Because base field is very small compared to extension field
- Hard part is individual logarithms
  - Usual descent algorithm not good enough
  - Need to be completed by new descent algorithm
  - Resulting complexity is:

$$L(1/\cancel{4}+o(1)).$$

  - Practical application:
    - New records in $\mathbb{F}_{2^{1778}}$, $\mathbb{F}_{2^{4080}}$ and $\mathbb{F}_{2^{6168}}$ recently announced
    - Other records by Göloğlu, Granger, McGuire and Zumbrägel

- $1778 = 2 \cdot 7 \cdot (2^7 - 1)$, 220 CPU-hours
- $4080 = 2 \cdot 8 \cdot (2^8 - 1)$, 14100 CPU-hours
- $6168 = 3 \cdot 8 \cdot (2^8 + 1)$ 550 CPU-hours

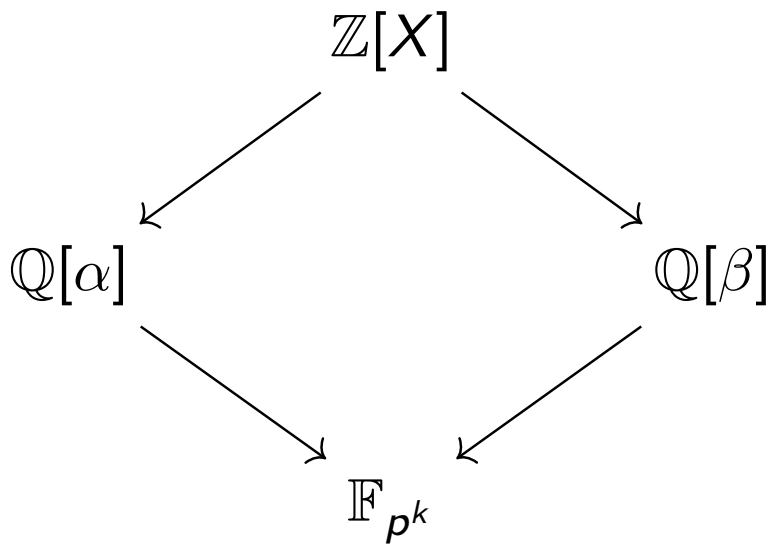
Resulting Complexity



Number field sieve for extension fields [JLSV06]

- Finite field of the form $\mathbb{F}_{p^k}$ with p larger than $L_Q(1/3)$
- Find two univariate polynomials f_1 and f_2
  - Such that $\gcd(f_1, f_2) \bmod p$ is irreducible of degree k
- This defines the finite field $\mathbb{F}_{p^k}$ as the smallest field where f_1 and f_2 have common roots.

Commutative diagram



where α and β resp. denote roots of f_1 and f_2 in $\bar{\mathbb{Q}}$.

Parameters that govern the complexity

- Degree and coefficient size of f_1 and f_2
- Degree of the elements of $\mathbb{Q}[X]$ we sieve on
  - When p is larger than $L_Q(2/3)$, sieve on linear polynomials
  - When p is between $L_Q(1/3)$ and $L_Q(2/3)$, need higher degree
    - $\Rightarrow$ larger constant in complexity

Improving some cases

- Joint work with Cécile Pierrot

- Find better polynomials when p is **special**
- Example: **pairing-based constructions** (MNT, BN)
- Setting:

$$p = h(u),$$

with h of moderate degree and small coefficients and u small (compared to p)

The case p in $[L_Q(1/3), L_Q(2/3)]$

- In [JLSV06]:
  - f_1 is an irreducible of degree k and $f_2 = f_1 + p$
  - Degrees are small, coeff. in f_2 are large

- New SNFS construction with $p = h(u)$
  - Choose $f_1(x) = f^{(0)}(x) - u$,
    with $f^{(0)}$ of degree k and small coeffs.
  - Then let $f_2 = h(f^{(0)})$, degree is $k \deg h$ and small coeffs.
- Trades smaller coefficients for higher degree, improves the complexity:

$$L_Q\left(1/3, \left(\frac{128}{9}\right)^{1/3}\right) \Rightarrow L_Q\left(1/3, \left(\frac{128}{9} \cdot \frac{\deg(h)+1}{2\deg h}\right)^{1/3}\right)$$

The case p in $[L_Q(2/3), Q]$

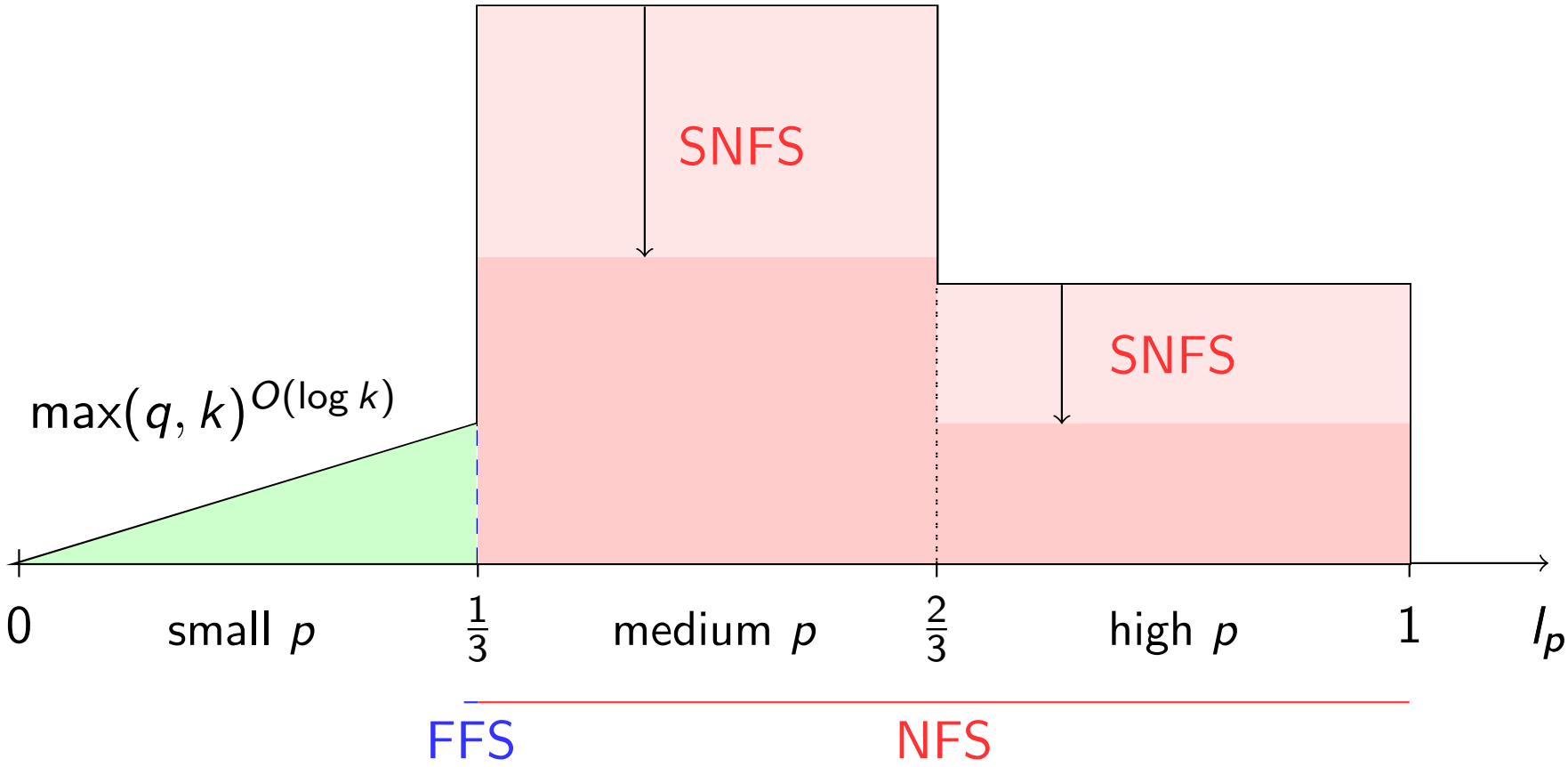
- In [JLSV06]:
  - Lattice reduction based construction

- New SNFS construction with $p = h(u)$ remains as before
- Complexity improvement is as in classical SNFS:

$$L_Q\left(1/3, \left(\frac{64}{9}\right)^{1/3}\right) \Rightarrow L_Q\left(1/3, \left(\frac{32}{9}\right)^{1/3}\right),$$

assuming that h has the right degree for the specific p and k balance

Resulting Complexity



Open problems

- Large characteristic (general case) ?
- Factoring ?
- Elliptic curves ?

Questions ?

Descent phase (option 1)

- In practice, bootstrap using continued fractions
- Classical descent (for high to mid degrees):
  - Consider $F(X, Y)$ of low degree in X and Y; let $r \approx \ell/2$
  - We have:

$$(F(X, X^{2^r}))^{2^{\ell-r}} = F^* \left(X^{2^{\ell-r}}, \frac{h_0(X)}{h_1(X)} \right)$$

  - To apply descent to f, find F such that $f | F(X, X^{2^r})$
- New descent (for mid to low degrees):
  - Find k_1 and k_2 such that $f | k_1^* \left(\frac{h_0(X)}{h_1(X)} \right) k_2(X) - k_1(X) k_2^* \left(\frac{h_0(X)}{h_1(X)} \right)$
  - Gives relation between above polynomial and

$$k_1(X)\, k_2(X) \prod_{\mu \in \mathbb{F}_{2^\ell}^*} (k_1(X) - \mu k_2(X))$$

Descent phase (option 2)

- Joint work with Barbulescu, Gaudry, Thomé
- Without Gröbner bases
- Improved complexity:

$$\exp(O(\log q \log k)).$$

- Sub-exponential but not practical (yet)
- Basic idea, evaluate $X^q - X$ at homography in $P(x)$ and 1

Index calculus: multiplicative generator ?

- After linear algebra:
  - Obtain an element of the kernel of equations
  - Modulo each (large) factor of group order
  - Discrete logarithms in any basis g_0 is a possible solution

- Conversely, **If**:
  - Matrix has "full" rank modulo each factor
  - And smoothness basis generates multiplicative group
- Then:
  - There exists g_0 such that kernel vector is Dlogs.
  - Moreover: any invertible entry in vector corresponds to a group generator.

- Alternative option that checks conditions:
  - Use different linear algebra (Smith normal form)
  - Proposed by Huang and Narayanan in *Finding Primitive Elements in Finite Fields of Small Characteristic.* [arXiv]